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Thermal entanglement between non-nearest-neighbor spins on fractal lattices

Yu-Liang Xu, Lu-Shun Wang and Xiang-Mu Kong*

Shandong Provincial Key Laboratory of Laser Polarization and Information Technology,

Department of physics, Qufu Normal university, Qufu 273165, China

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Abstract

We investigate thermal entanglement between two non-nearest-neighbor sites in ferromagnetic Heisenberg chain and on fractal lattices by means of the decimation renormalization-group (RG) method. It is found that the entanglement decreases with increasing temperature and it disappears beyond a critical value T_c. Thermal entanglement at a certain temperature first increases with the increase of the anisotropy parameter Δ and then decreases sharply to zero when Δ is close to the isotropic point. We also show how the entanglement evolves as the size of the system L becomes large via the RG method. As L increases, for the spin chain and Koch curve the entanglement between two terminal spins is fragile and vanishes when $L \geq 17$, but for two kinds of diamond-type hierarchical (DH) lattices the entanglement is rather robust and can exist even when L becomes very large. Our result indicates that the special fractal structure can affect the change of entanglement with system size.

*Corresponding author; Electronic address: kongxm@mail.qfnu.edu.cn (X.-M. Kong)

I. INTRODUCTION

An essential difference between quantum and classical physics is the possible existence of nonlocal correlation in quantum system which is called the entanglement [1]. Recently, the quantum entanglement has been recognized as an crucial resource in various fields of quantum information such as quantum communication and computation [2–7]. Since the entanglement is fragile and sensitive to many environment factors, many efforts are devoted to studying stable entanglement for realistic system in finite temperature. Thus, thermal entanglement has naturally received much attention by its advantage of stability and requiring neither measurement nor controlled switching of interactions in the preparing process. For spin systems can be used for gate operation in quantum computer thermal entanglement on solid spin systems have been widely studied, for example, the Heisenberg spin chain both in the absence [8] and presence [9, 10] of an external magnetic field , spin rings [11] and spin clusters [12, 13]. However, most of these works only focused on thermal entanglement between nearest, next-nearest or next-to-next-nearest neighbor spins [14–19]. This motivate us to propose two questions: (i) Can thermal entanglement exist between distant non-nearest-neighbor sites in spin system? (ii) How does thermal entanglement evolve as system size grows? But it is very difficult to obtain exact results on entanglement in spin systems on arbitrary lattices especially fractal lattices, since this usually requires the expression of the partition function which is too complicated to solve when the system size becomes very large.

In recent years, the entanglement at zero temperature in the large size system has been studied by adopting the renormalization-group (RG) method. In 2002, A. Osterloh *et al* first introduced the density matrix renormalization-group (DMRG) approach to study the entanglement close to the quantum phase transition (QPT) [20] and reveal a profound difference between classical correlations and the non-local quantum correlation. Further,by applying the quantum renormalization-group (QRG) approach, M. Kargarian *et al* investigated the entanglement in the anisotropic Heisenberg model [21, 22] and discussed the nonanalytic behaviors and the scaling close to the quantum critical point of the system. Recently We have calculated the block-block entanglement in the XY model without and with staggered Dzyaloshinskii-Moriya (DM) interaction by using this QRG method and have found the DM interaction can enhance the entanglement and influence the QPT of the system [23, 24].

Inspired by above idea, we apply the real-space renormalization-group (RSRG) approach which is developed in the Refs. [25–35] to study the thermal entanglement between two end sites in the spin chain, Koch curve and on the diamond-type hierarchical (DH) lattices and analyze the influence of the temperature, the anisotropy parameter and the system size on the entanglement.

II.   MODEL AND METHOD

The effective Hamiltonian of the spin-1/2 anisotropic ferromagnetic Heisenberg spin chain with L sites is

$$- \beta H = \sum_{i}^{L} K \left[(1 - \Delta) \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + \sigma_i^z \sigma_{i+1}^z \right], \tag{1}$$

where σ_i^α ($\alpha = x, y, z$) denote the Pauli operators at site i. $K = \beta J = J/k_B T$, $J > 0$ is the exchange coupling parameter, and k_B is the Boltzmann constant. For simplicity, we assume that $k_B = 1$ and $J = 1$. The sum is over all the nearest-neighbor spin pairs and $\Delta \in (-\infty, 1]$ is the anisotropy parameter. For $\Delta = 0$ and $\Delta = 1$, the isotropic Heisenberg (XXX) and Ising model are obtained, respectively. The state of the above system in thermal equilibrium can be described by the density operator $\rho = Z^{-1} \left(e^{-\beta H} \right)$, where $Z = \mathrm{Tr} \left(e^{-\beta H} \right)$ is the partition function.

The entanglement of two-qubit system in the thermal state ρ_{12} can be calculated by the negativity [36] which is based on the partial transpose method [37]. The negativity N is defined as

$$N \left(\rho_{12} \right) = 2 \sum_{i} |\mu_i|, \tag{2}$$

where μ_i is the negative eigenvalue of $\rho_{12}^{T_1}$, and T_1 denotes the partial transpose with respect to the first subsystem. According this definition one can easily obtain thermal entanglement $N \left(K', \Delta' \right)$ of the two-spin chain with the Hamiltonian $H'_{12} \left(K', \Delta' \right)$ (such as Fig 1. (a) $n = 0$ shown). However, when the size of system becomes large, the density matrix is difficult or impossible to gain. The entanglement of two terminal spins on this system can not be directly worked out.

We apply the decimation RSRG method to solve the above problem. This decimation RSRG method [27, 29, 38] has proved to be successful in spin chain and especially the fractal

lattices. For the spin chain, this decimation procedure is illustrated in Fig. 1 (a). Simply, the generator is taken out in the infinite system. The generator with the Hamiltonian $H_{132}(K, \Delta)$ is renormalized into the new two-site chain with the Hamiltonian $H'_{12}(K', \Delta')$ by integrating the internal site 3 with the partition function being preserved. This transformation can be described as

$$\exp(H'_{12}) = \text{Tr}_3 \exp(H_{132}). \tag{3}$$

We can obtain the recurrence relation between the original parameters (K, Δ) and the new parameters (K', Δ') by solving the trace Tr_3 with the method developed in Refs [29, 31]. Combining this relation $K' = g(K, \Delta)$, $\Delta' = h(K, \Delta)$ and the negativity $N(K', \Delta')$ of two-site system, the entanglement between two terminal spins in three-qubit chain can be obtained as follow

$$N(K', \Delta') = N(g(K, \Delta), \ h(K, \Delta)). \tag{4}$$

The entanglement between two distant terminal spins in the Heisenberg chain can be calculated after many iterations of the recurrence relation. We take use of the same method to study thermal entanglement between two terminal sites in Koch curve and on the DH lattices as shown in Fig. 1 (b) (c) and (d). The analytical expression about the negativity in Eq. (4) is difficult to obtain, we will show some numerical results.

III. HEISENBERG SPIN CHAIN

We first study how the entanglement between terminal sites in spin chain with different number of sites L varies with temperature T at $\Delta = -0.2$ (shown in Fig. 2). For different cases of L, the results have the similar feature that the entanglement decreases monotonically with increasing temperature and it vanishes beyond the critical temperature T_c. At $T = 0$, the system is in the entangled ground state. As temperature increases, the entanglement decreases due to the mixture of the unentangled excited state with the ground state. At $T = T_c$, the system is governed by the unentangled excited state completely, and therefore the entanglement vanishes. Comparing the entanglement of terminal sites in the different system, it is found that the entanglement of the ground state decreases sharply with increasing L. Different from the maximally entangled Bell ground state for $L = 2$ system, the ground state for $L > 2$ system becomes a degenerate and related to Δ state which cause

the decrease of entanglement. The energy gap between the ground and the unentangled excited state increases with little range but the thermal energy of system increases with large range when L increases. For $L > 2$, the system can easily overcome the gap and enter the unentangled state. That leads to the decrease of T_c. This phenomenon reflects that thermal fluctuation of internal sites may suppress quantum effect.

The influence of the anisotropic parameter Δ on the entanglement between two terminal sites at a fixed temperature $T = 0.01$ is plotted in Fig. 3. As can be seen, for $L = 2$ case, the system firstly keeps in the ground state with the entanglement $N = 1$. Then, as Δ approaches the isotropic point $\Delta = 0$, the energy gap between the ground and the unentangled excited states becomes so small that the system can jump to the unentangled excited state. Therefore there exists a sharp decease for entanglement and the entanglement vanishes when Δ is close to zero. This result also accords with that the entanglement can not exist in an isotropic Heisenberg ferromagnetic chain in Ref [39]. For $L \neq 2$ cases, it is found that the entanglement increases firstly with increasing Δ because the ground state is related to Δ and will change with Δ. All entanglement jump down to zero when Δ reaches to zero. From above results, we can see that the entanglement is fragile and when $L \geq 17$ the entanglement does not exist whatever the temperature and the anisotropic parameter are.

IV.   FRACTAL LATTICES

The properties of phase transition on different fractal lattices have been studied by the RSRG method, and the entanglement on these self-similar lattices remains to be explored. We turn to the study of the entanglement between end sites in Koch curves with non-integer fractal dimension $d_f = \ln 4 / \ln 3$ and plot the numerical results of negativity versus T and Δ for different L in Fig. 4. Compared with the entanglement in the spin chain ($L = 5$), it has similar properties that the entanglement decreases with T and the maximal of entanglement and T_c are approximately equal. But the entanglement variation versus Δ is very different from that in spin chain, i.e., the range that the entanglement can exist is smaller, the maximal of entanglement is lager. The entanglement decreases quickly as L becomes large and there exists no entanglement any longer when $L \geq 17$. We can deduce this result from the similar Hamiltonian and open boundary conditions of these two systems.

Now we consider two kinds of DH lattices with fractal dimensions $d_f = 2$ (lattice A, for simplicity) and $d_f = \ln 5/\ln 2$ (lattice B, for simplicity). The RG transformation on these DH lattices respectively have been shown in Fig. 1 (c) and Fig. 1 (d). We first discuss the dependence of the entanglement between terminal sites on T with $\Delta = -0.2$. In Fig. 5, one can find some similar behaviors that the entanglement is the maximal value at $T = 0$, and the entanglement decreases with increasing temperature and vanishes beyond the critical temperature. However, some different phenomenons are also observed that the entanglement between end sites on these two lattices decrease more slowly with increasing L and it still exists even though the system size becomes very large ($L = 1564$). For the case of on lattice B, as L increases, the entanglement at zero temperature decreases but the corresponding T_c increases. It is obvious that the entanglement for different L crosses at $T \approx 0.49$. This result indicates that these two fractal lattices have special energy level structure. The energy gap between the entangled ground state and the unentangled excited state is so large that the system can jump to the unentangled excited state only at higher T.

The variation of the entanglement between end sites on DH lattices versus Δ at $T = 0.01$ is also discussed. As can be seen in Fig. 6 (a) for the lattice A, the entanglement firstly increases as Δ increases, and then it quickly decays to zero when Δ reaches the isotropic point. The entanglement decrease very slowly when L becomes very large and there exist no cross point when Δ reaches zero. For the case of lattice B, Fig. 6 (b) shows that the entanglement also exhibits stable and it changes very little when Δ is not very close to zero. The entanglement mainly remains robust with the increase of L. In this graph, we also observe that a "entanglement crossing" occurs at $\Delta \approx -0.045$. At a fixed temperature, the thermal excited energy of the system is determined. Only when Δ is very close to zero, the energy gap between the entangled ground state and the unentangled excited state can become so small that the system can enters the unentangled state. It also indicates that the different fractal structure can influence the entanglement by changing the energy level structure of system.

V.   CONCLUSION

We have investigated thermal entanglement between two end spins in Heisenberg chain, Koch curve and on two kinds of DH lattices with $d_f = 2$ and $d_f = 2.32$ by the decimation RG method. The effect of the temperature and the anisotropy parameter on thermal entanglement is discussed. It is found that the symmetry of system and the thermal fluctuation can suppress or promote the quantum effect at different conditions. We also have noticed that the entanglement on some special lattices may exhibit different property when the system size L becomes large. The entanglement on two kinds of DH lattices is quite robust and it can survive even though L becomes very large in contrast to that in spin chain. The phenomenon of the "entanglement crossing" indicates that the special fractal structure does influence on the entanglement.

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Figure captions:

Fig. 1. The procedure of the RG transformation. From (a) to (d), it shows the transformation of one-dimensional spin chain, Koch curve, the diamond-type hierarchical lattice with fractal dimension $d_f = 2$ and $d_f = \ln 5 / \ln 2$.

Fig. 2. The entanglement between two end sites in Heisenberg ferromagnetic chain versus temperature at $\Delta = -0.2$ for different number of sites L (from top to bottom, $L = 2, 3, 5$ and 9).
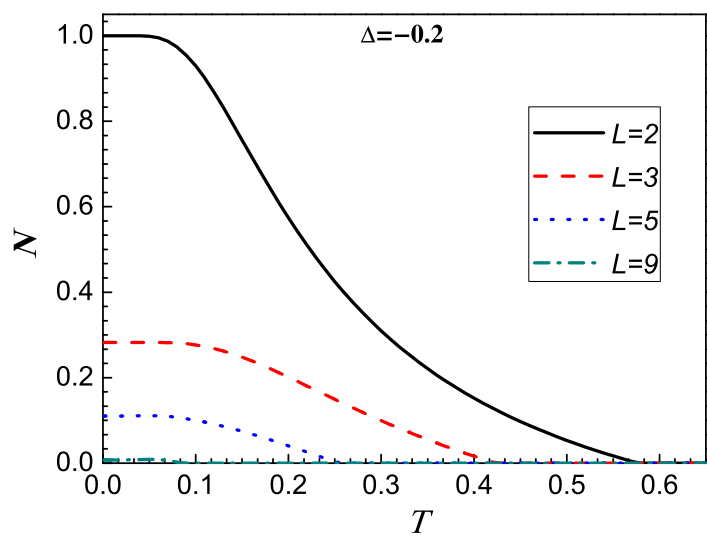
Fig. 3. The entanglement between two end sites in Heisenberg ferromagnetic chain as anisotropy parameter Δ at $T = 0.01$ for different number of sites L (from top to bottom, $L = 2, 3, 5$ and 9).
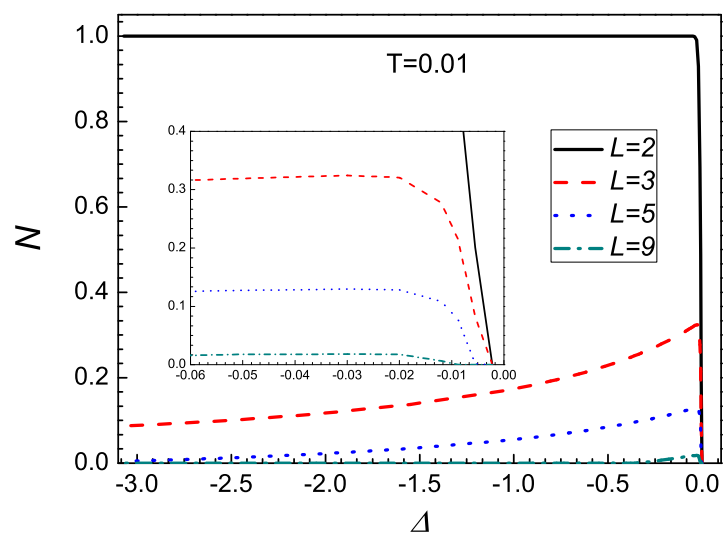
Fig. 4. The variation of the entanglement between end sites in Koch curve: (a) the entanglement versus temperature T at $\Delta = -0.2$. (b) the entanglement versus Δ at $T = 0.01$.
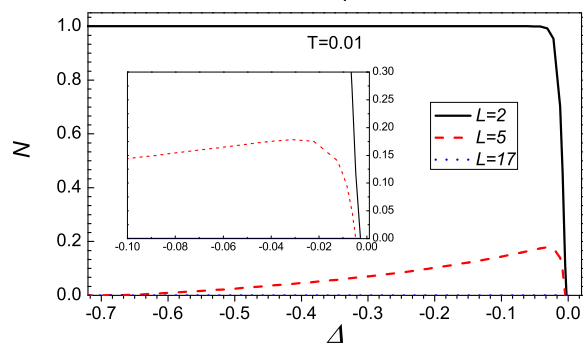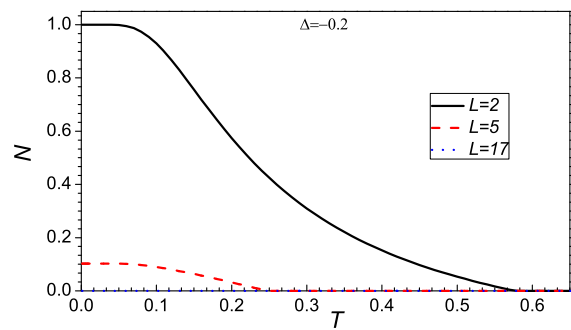
Fig. 5. Upper panel: the negativity of two end sites on the DH lattice with $d_f = 2$ versus T at $\Delta = -0.2$. Lower panel: the negativity of two end sites on the DH lattice with $d_f = \ln 5 / \ln 2$ versus T at $\Delta = -0.2$. The negativity for different value of L has a cross point at $T \approx 0.49$.
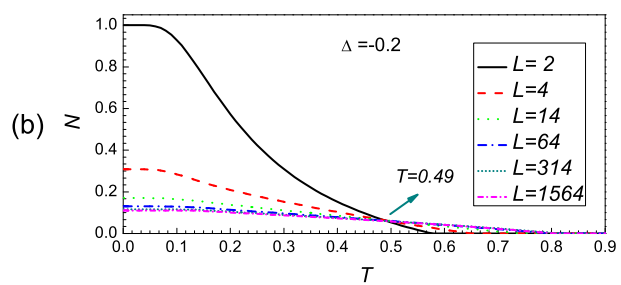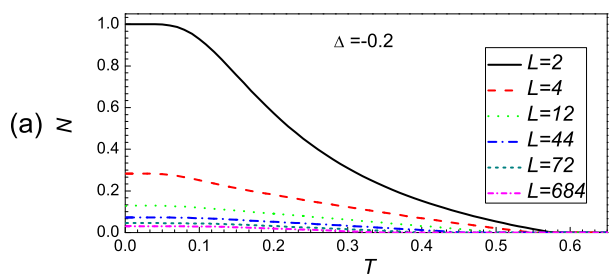
Fig. 6. Upper panel: the evolution of the entanglement between terminal sites on DH lattice with $d_f = 2$ as Δ increases at $T = 0.01$ for different value of L. Lower panel: the entanglement between terminal sites on DH lattice with $d_f = \ln 5 / \ln 2$ versus Δ at $T = 0.01$ for different value of L. The entanglement for different L has a cross point at $\Delta \approx -0.045$.

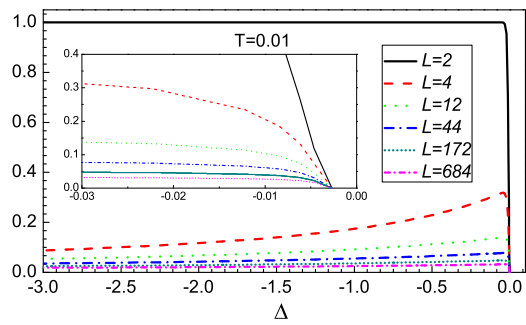**1**

3

2

RG

1

2

(a)

n=1

n=0

1

3

4

5

2

RG

1

2

(b)

n=1

n=0

1

3

4

2

RG

1

2

(c)

1

3

4

2

RG

1

2

(d)

Δ=−0.2

T=0.01

(a)

N

$\Delta = -0.2$

——	$L=2$
- - -	$L=4$
······	$L=12$
—·—	$L=44$
······	$L=72$
—·—	$L=684$

(b)

N

$\Delta = -0.2$

$T=0.49$

——	$L=2$
- - -	$L=4$
······	$L=14$
—·—	$L=64$
······	$L=314$
—·—	$L=1564$

T

(a)

(b)